EXPANSIONS ON A GEOMETRIC PROOF OF THE IRRATIONALITY OF THE SQUARE ROOT OF TWO

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ABSTRACT. A number of expansions to the geometric proof of the irrationality of the square root of two have been put forward in the paper “Irrationality From The Book” by Steven J. Miller and David Montague. There are a number of other conceptually simple expansions which can be attempted. This paper shows some basic work towards proving the irrationality of \( \sqrt{p} \), where \( p \) is a prime, and of \( \sqrt[3]{2} \). Although no final proofs are presented, preliminary work is shown which might help future mathematicians to find a geometric proof of these facts, or at least avoid some dead ends.

INTRODUCTION

In class we read through a geometric proof of the irrationality of \( \sqrt{2} \). Although there are many proofs for this fact, the geometric one is relatively new and particularly elegant [1]. The starting point for this proof is the assumption that \( \sqrt{2} \) is rational, that is that it can be expressed as a quotient of two integers which are the smallest possible. By doing some algebra,

\[
\frac{a}{b} = \sqrt{2}
\]

becomes

\[
a^2 = 2b^2.
\]

This means that we can find some smallest integers for which the square of one is equal to twice the square of the other. The geometric equivalent of this equation is that inside a large square with side length \( a \) two smaller squares with side length \( b \) can be perfectly fit. Looking at the figure below, we can see that if that were the case, then the double-counted portion in the middle of the figure would need to be equal to the sum of the two uncovered portions [2].

We can see that the double-counted portion is a square with side length \( 2b - a \). Additionally, the two uncounted portions are both squares with side length \( a - b \). If these are equal to each other, then we have a new equation which is equivalent to \( \sqrt{2} \), namely

\[
\frac{2b - a}{a - b}.
\]
$2b - a$ and $a - b$ are smaller than $a$ and $b$ respectively, so this contradicts the notion that $\frac{a}{b}$ was the smallest possible fractional representation of $\sqrt{2}$. Thus $\sqrt{2}$ cannot be expressed as a quotient and therefore is irrational [2].

The paper “Irrationality From The Book” continues on to use the same argument to show that $\sqrt{3}$ is irrational. Instead of using squares, the paper uses equilateral triangles because the area of an equilateral triangle is proportional to the square of the length of its sides. By placing three smaller triangles inside the larger triangle a similar argument can be made for the the irrationality of $\sqrt{3}$ [3].

1. Expansion to Pentagons

The same paper makes an argument for the irrationality of $\sqrt{5}$ using regular pentagons, as the area of a regular pentagon is proportional to the square of the length of its sides. By placing five regular pentagons inside one larger pentagon, we can attempt to use a similar argument to show that the uncounted and double-counted portions of the figure represent a smaller rational representation of $\sqrt{5}$. The problem with this proof is that there are additional uncounted areas besides the center regular pentagon in the form of five triangles between the bases of each pair of pentagons. Additionally, the double-counted portions don’t appear to be regular pentagons [4].

However, the authors of the paper find a trick to solve these problems. By finding an equivalent triangle in the double-counted portions for each uncounted triangle, we can cancel out these areas and are left with five pentagons, which are shown to be regular, and the center uncounted regular pentagon. Now we have the ingredients to show that there is a smaller quotient which equals $\sqrt{5}$, so therefore it cannot be a rational number [4].

2. Attempt at Expansion to Heptagons

For this project I hoped to expand this method to show that the square root of any prime is irrational. To start with, I attempted to apply the methods used in the proof of the irrationality of $\sqrt{5}$ to show that $\sqrt{7}$ was likewise irrational. Theoretically, there would be similar uncounted and double-counted areas, which could then be divided to show that a smaller representation of the $\sqrt{7}$ could be formed. Although the double-counted portions would not look like heptagons initially, by eliminating an equivalent triangle, as shown with pentagons, we could create heptagons in each of the seven double-counted areas.
Unfortunately, the actual application of this trick did not work out nearly as well. Although the uncounted triangles were again created, the equivalent triangles are actually too large. Instead of neatly cutting the double-counted area into a triangle and a regular heptagon, as in the pentagon case, it extended beyond the first vertices of the double-counted area, leaving an oddly shaped pentagon and two small additional triangles outside of the double-counted portion. Look below for a figure showing this attempt.

Although after computing the area of all the different sections it was confirmed that the uncounted portions and double-counted portions would cancel each other out, it was not apparent that there was a smaller rational representation of $\sqrt{7}$.

The root of the problem lay in the length of the sides of the uncounted triangles. Looking at the triangle at the base, as represented in the figure on the next page, we know that the length of the base of the triangle is $1 - \frac{2}{\sqrt{7}}$ if we have the length of the sides of the large heptagon be 1 and the length of the sides of the smaller heptagons be $\frac{1}{\sqrt{7}}$. We also know that for a regular heptagon the interior angles will all be equal to $\frac{5\pi}{7}$. Because the interior angle of the triangle at the base will be the complement of that angle, we know it to be $\frac{2\pi}{7}$. Now that we have an angle and the length of a side we can use trigonometry to find the length of the other sides of the triangle. It turns out that the length of either of the other sides is $\approx 0.19573$. The equivalent triangle will have sides of equal length, so in total the length of the straight line formed by the opposite sides of the two triangles is $\approx 0.39146$. This line follows a single side of one of the smaller regular heptagons. Recall that each side length is $\frac{1}{\sqrt{7}}$, which is $\approx 0.377964$. Therefore, we can see that the equivalent triangle will extend beyond the first side of the regular heptagon. In the pentagon case, this did not happen. Instead, the equivalent triangle extended far enough to create a regular pentagon in
the remaining double-counted portion. In the heptagon case, an irregular pentagon was created instead of a regular heptagon, so the same argument cannot be made.

3. Further Primes

Originally we hoped to expand some of the methods laid out earlier to show that the square root of any prime was irrational. However, as we can see from the case of $\sqrt{7}$, this doesn’t work, at least not for every prime. Now I hope to show that further primes will run into the same problems. For every further prime, the length of the sides of each smaller regular polygon will be $\frac{1}{\sqrt{p}}$. As $p$ continues to grow, the length of the individual sides will shrink. However, the length of the sides of the large regular polygon will stay at 1, so the length of the base of the uncounted triangle will grow. Although the interior base angle of the triangle will decrease, it will not decrease quickly enough to offset the increase base length. Therefore, the sides of the equivalent triangle will continue to extend past the length of the individual side of the smaller regular polygon, which will continue to prevent us from showing a smaller rational expression for the square root of the prime in question.

4. Attempt at Expansion to Cubics

In addition to larger primes, I also attempted to extend the methods shown earlier to the cube root of two. My thinking was that whereas the original geometric proof used two overlapping squares within a larger square, I would use two overlapping cubes inside a larger cube. By enumerating the size of the double-counted and uncounted areas I could find a smaller rational expression for the cube root of two, and therefore prove its irrationality. Like the expansion to larger primes, this ended up leading, at least as far as I could tell, to a dead end.

The reason that this was a dead end was due to the uncounted space in the geometric construction. What would be desired would be two cubes which are uncounted and cancel out the single double-counted cube in the center of the figure. Unfortunately, this does not appear. Instead, we are left with six small cubes and six rectangular solids, as shown in the figure on the next page. If three cubes and three rectangular solids could be combined to create one cube, there would be hope. However, there doesn’t appear to be a way to do this because of the dimensions of the rectangular solids. Intellectually this makes sense because the addition of a third dimension eliminates the neatness of the overlapping shapes.

If ones goes even further and thinks about how to show that the cube root of three is irrational, then it becomes more obvious why a third dimension would wreck havoc upon the geometric method. If one
were to consider a triangular pyramid and attempt to place three smaller triangular pyramids inside, it
wouldn’t even be readily apparent how to overlap the pyramids, as there are four vertices in the pyramid,
and only three will have pyramids placed into them. Although perhaps some future work might be done
which demonstrates how to expand the geometric proof into the third (or even fourth) dimension, I haven’t
discovered a method during my studies so far.

References

[2] Ibid., 2.
5. Notes

• What other methods of proving the irrationality of $\sqrt{2}$ exist?
  See [5] for a list of a bunch.

• When two cubes are overlapped in a larger cube, what spaces haven’t been counted?
  See section 4 for my answers and what I worked through.

• Can we combine the extra spaces in the large cube into a smaller cube either geometrically or algebraically?
  Not as far as I can tell.

• Are the double-counted areas regular polygons once the triangle is removed?
  No, as the side length of the triangle will sometimes be too large. Also, the angles formed by the small polygons and their intersections vary, so regularity is out of the question.

• How big must the smaller polygons be to get the true regular polygon in the uncounted center of the figure?
  Solve the equation
  \[ a^2 = pb^2 \]
  for whatever $p$ you are working with and an $a$ of 1. The result is therefore
  \[ b = \frac{1}{\sqrt{p}}. \]

• Despite the lack of nice regular polygons in the double-counted sections and the extra triangles created by the uncounted triangle cancelling out effort, is the area of the $p$ $p$-gons still equal to the area of the larger $p$-gon?
  Yes, despite any lack of geometric proof of irrationality, the areas still more or less cancel out.